Elements of dynamical systems for bifurcation analysis applied to biology

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Plan

The classical mathematical framework

ODE system :

$$\frac{dx_1}{dt} = f_1(x_1, x_2) \qquad \frac{dx_2}{dt} = f_2(x_1, x_2) \tag{1}$$

Definition

We say $(\bar{x}_1, \bar{x}_2)$ is an equilibrium point if $f_1(\bar{x}_1, \bar{x}_2) = 0$ and $f_2(\bar{x}_1, \bar{x}_2) = 0$.

Definition

The Jacobian matrix at $\bar{x} = (\bar{x}_1, \bar{x}_2)$ associated with the system (1) is the following matrix : $Jac(f, \bar{x}) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(\bar{x}_1, \bar{x}_2) & \frac{\partial f_1}{\partial x_2}(\bar{x}_1, \bar{x}_2) \\ \frac{\partial f_2}{\partial x_1}(\bar{x}_1, \bar{x}_2) & \frac{\partial f_2}{\partial x_2}(\bar{x}_1, \bar{x}_2) \end{pmatrix}$

Rmk : looking at the Jacobian matrix is equivalent to look at the linearized problem.

Definition

We call eigenvalues of the matrix $Jac(f,\bar{x})$*, the solutions of the equation*

$$\begin{vmatrix} \frac{\partial f_1}{\partial x_1}(\bar{x}_1,\bar{x}_2)-\lambda & \frac{\partial f_1}{\partial x_2}(\bar{x}_1,\bar{x}_2) \\ \frac{\partial f_2}{\partial x_1}(\bar{x}_1,\bar{x}_2) & \frac{\partial f_2}{\partial x_2}(\bar{x}_1,\bar{x}_2)-\lambda \end{vmatrix} = \lambda^2 - B\lambda + C = 0 \textit{ with :}$$

$$B = \frac{\partial f_1}{\partial x_1}(\bar{x}_1,\bar{x}_2) + \frac{\partial f_2}{\partial x_2}(\bar{x}_1,\bar{x}_2)$$

$$C = \frac{\partial f_1}{\partial x_1}(\bar{x}_1,\bar{x}_2)\frac{\partial f_2}{\partial x_2}(\bar{x}_1,\bar{x}_2) - \frac{\partial f_2}{\partial x_1}(\bar{x}_1,\bar{x}_2)\frac{\partial f_1}{\partial x_2}(\bar{x}_1,\bar{x}_2)$$

Theorem

1. *if* $Re(\lambda_1) < 0$, $Re(\lambda_2) < 0$, *then the linearized system is stable and the* $\bar{x}$ *is said to be asymptotically locally stable.*
2. *if* $Re(\lambda_1) > 0$ *or* $Re(\lambda_2) > 0$, *then the linearized system is unstable and the* $\bar{x}$ *is said to be asymptotically locally unstable.*
3. *if* $Re(\lambda_1) = 0$ *or* $Re(\lambda_2) = 0$, *other techniques have to be used to determine the stability.*

An example from Schmeiser [2020]

ODE system :

$$\frac{dc_M}{dt} = f(c_P) - \gamma_M c_M$$
$$\frac{dc_P}{dt} = \alpha_M c_M - \gamma_P c_P \qquad (2)$$

The equilibrium point is $\bar{c} = (\bar{c}_M, \bar{c}_P)$ with $\bar{c}_M = \frac{\gamma_P \bar{c}_P}{\alpha_M} = \frac{f(\bar{c}_P)}{\gamma_M}$

The Jacobian matrix is $Jac(f, \bar{c}) = \begin{pmatrix} -\gamma_M & f'(\bar{c}_P) \\ \alpha_M & -\gamma_P \end{pmatrix}$

The eigenvalues are solutions of :

$$\lambda^2 + (\gamma_M + \gamma_P)\lambda - f'(\bar{c}_P)\alpha_M + \gamma_M\gamma_P = 0$$
$$\Delta = (\gamma_M + \gamma_P)^2 + 4f'(\bar{c}_P)\alpha_M - 4\gamma_M\gamma_P < (\gamma_M + \gamma_P)^2$$

which are :

$$\lambda_{1,2} = \frac{-(\gamma_M + \gamma_P) \pm \sqrt{\Delta}}{2} \quad s.t. \quad Re(\lambda_{1,2}) < 0$$

Hence, $\bar{c}$ is locally asymptotically stable.

Another example : Hopf bifurcation from Schmeiser [2020]

ODE system

$$\frac{dc_1}{dt} = f_2(c_2(t-\tau_1)) - \gamma_1 c_1(t) \qquad \frac{du}{dt} = f_2(v) - \gamma_1 u$$
$$\frac{dc_2}{dt} = f_1(c_1(t-\tau_2)) - \gamma_2 c_2(t) \qquad \frac{dv}{dt} = f_1(u(t-\tau)) - \gamma_2 v$$

Equilibrium point $\bar{x} = (\bar{u}, \bar{v})$ s.t. $\bar{u} = \frac{f_2(\bar{v})}{\gamma_1}$ and $\bar{v} = \frac{f_1(\bar{u})}{\gamma_2}$
We cannot look at the Jacobian matrix. We have to linearize the problem.

$$\frac{du}{dt} = f_2'(\bar{v})v - \gamma_1 \bar{u}$$
$$\frac{dv}{dt} = f_1'(\bar{u})u(t-\tau) - \gamma_2 \bar{v}$$

We have only information about $f_2'(\bar{v})$ and $f_1'(\bar{u})$

We make the Ansatz $u(t) = u_0 e^{\lambda t}$ and $v(t) = v_0 e^{\lambda t}$, so that we obtain :

$$\frac{du}{dt} = f_2'(\bar{v})v - \gamma_1 \bar{u}$$
$$\frac{dv}{dt} = f_1'(\bar{u})ue^{-\lambda\tau} - \gamma_2 \bar{v} \tag{3}$$

We can compute the Jacobian matrix of (3) :
$Jac(f, \bar{x}) = \begin{pmatrix} -\gamma_1 & f_2'(\bar{v}) \\ f_1'(\bar{u})e^{-\lambda\tau} & -\gamma_2 \end{pmatrix}$
Eigenvalues are solutions of :

$$\lambda^2 + \lambda(\gamma_1 + \gamma_2) + \gamma_1\gamma_2 - f_2'(\bar{v})f_1'(\bar{u})e^{-\lambda\tau} = 0$$

$$\lambda^2 + \lambda(\gamma_1 + \gamma_2) + \gamma_1\gamma_2 - f_2'(\bar{v})f_1'(\bar{u})e^{-\lambda\tau} = 0$$

$$\begin{aligned} \Delta &= (\gamma_1 + \gamma_2)^2 - 4(\gamma_1\gamma_2 - f_2'(\bar{v})f_1'(\bar{u})e^{-\lambda\tau} \\ &= \gamma_1^2 + \gamma_2^2 + 4f_2'(\bar{v})f_1'(\bar{u})e^{-\lambda\tau} \end{aligned}$$

$$\lambda_{1,2} = \frac{-(\gamma_1 + \gamma_2) \pm \sqrt{\Delta}}{2}$$

When $\tau = 0$, $\lambda_{1,2} < 0$, but when τ varies the sign of $Re(\lambda_{1,2})$ changes.

Other examples

- Widder et al. [2007]
- Tsumoto et al. [2006]

Bibliography

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